The Remainder and Factor Theorems

You factored quadratic expressions to solve equations.

Now

1 Divide polynomials using long division and synthetic division.

2 Use the Remainder and Factor Theorems.

Why?

The redwood trees of Redwood National Park in California are the oldest living species in the world. The trees can grow up to 350 feet and can live up to 2000 years. Synthetic division can be used to determine the height of one of the trees during a particular year.

New Vocabulary

- synthetic division
- depressed polynomial
- synthetic substitution

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**Example 1** Use Long Division to Factor Polynomials

Factor $6x^3 - 25x^2 + 18x + 9$ completely using long division if $(x - 3)$ is a factor.

\[
\begin{array}{c|cccc}
\text{x} & 6x^3 & -25x^2 & +18x & +9 \\
\hline 
3 & 6x^3 & -18x^2 & & \\
\quad & -25x^2 & +18x & & \\
\quad & -7x^2 & +18x & & \\
\quad & -7x^2 & +21x & & \\
\quad & & -3x & +9 & \\
\quad & & -3x & +9 & \\
\quad & & 0 & & \\
\end{array}
\]

Multiply divisor by $6x^2$ because $\frac{6x^3}{x} = 6x^2$.

Subtract and bring down next term.

Multiply divisor by $-7x$ because $\frac{-7x^2}{x} = -7x$.

Subtract and bring down next term.

Multiply divisor by $-3$ because $\frac{-3x}{x} = -3$.

Subtract. Notice that the remainder is 0.

From this division, you can write $6x^3 - 25x^2 + 18x + 9 = (x - 3)(6x^2 - 7x - 3)$.

Factoring the quadratic expression yields $6x^3 - 25x^2 + 18x + 9 = (x - 3)(2x - 3)(3x + 1)$.

So, the zeros of the polynomial function $f(x) = 6x^3 - 25x^2 + 18x + 9$ are $3, \frac{3}{2},$ and $-\frac{1}{3}$. The x-intercepts of the graph of $f(x)$ shown support this conclusion.

Guided Practice

Factor each polynomial completely using the given factor and long division.

1A. $x^3 + 7x^2 + 4x - 12; x + 6$

1B. $6x^3 - 2x^2 - 16x - 8; 2x - 4$
Long division of polynomials can result in a zero remainder, as in Example 1, or a nonzero remainder, as in the example below. Notice that just as with integer long division, the result of polynomial division is expressed using the quotient, remainder, and divisor.

Recall that a dividend can be expressed in terms of the divisor, quotient, and remainder.

\[
\text{divisor} \cdot \text{quotient} + \text{remainder} = \text{dividend}
\]

This leads to a definition for polynomial division.

### KeyConcept Polynomial Division

Let \( f(x) \) and \( d(x) \) be polynomials such that the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \) and \( d(x) \neq 0 \). Then there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x) \cdot q(x) + r(x),
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If \( r(x) = 0 \), then \( d(x) \) divides evenly into \( f(x) \).
Example 3  Division by Polynomial of Degree 2 or Higher

Divide $2x^4 - 4x^3 + 13x^2 + 3x - 11$ by $x^2 - 2x + 7$.

\[
\begin{array}{c|c}
2x^2 & -1 \\
\hline
x^2 - 2x + 7 & 2x^4 - 4x^3 + 13x^2 + 3x - 11 \\
& (-) 2x^4 - 4x^3 + 14x^2 \\
& \quad -x^2 + 3x - 11 \\
& \quad (-) -x^2 + 2x - 7 \\
& \quad \frac{\times}{x - 4}
\end{array}
\]

You can write this result as

\[
\frac{2x^4 - 4x^3 + 13x^2 + 3x - 11}{x^2 - 2x + 7} = 2x^2 - 1 + \frac{x - 4}{x^2 - 2x + 7}.
\]

Guided Practice

Divide using long division.

3A. $(2x^3 + 5x^2 - 7x + 6) \div (x^2 + 3x - 4)$

3B. $(6x^5 - x^4 + 12x^2 + 15x) \div (3x^3 - 2x^2 + x)$

Synthetic division is a shortcut for dividing a polynomial by a linear factor of the form $x - c$. Consider the long division from Example 1.

<table>
<thead>
<tr>
<th>Long Division</th>
<th>Suppress Variables</th>
<th>Collapse Vertically</th>
<th>Synthetic Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notice the coefficients highlighted in colored text. $6x^2 - 7x - 3$</td>
<td>Suppress $x$ and powers of $x$. $6 - 7 - 3$</td>
<td>Collapse the long division vertically, eliminating duplications. $-3\overline{6} - 25 + 18 + 9$</td>
<td>Change the signs of the divisor and the numbers on the second line. $3 \overline{\mid} 6 - 25 18 9$</td>
</tr>
<tr>
<td>$x - 3 \overline{\mid} 6x^3 - 25x^2 + 18x + 9$</td>
<td>$-3\overline{6} - 25 + 18 + 9$</td>
<td>$-1821 9$</td>
<td>$\overline{\mid} 18 - 21 - 9$</td>
</tr>
<tr>
<td>$(-) 6x^3 - 18x^2$</td>
<td>$(-) 6 - 18$</td>
<td>$6 - 7 - 3 0$</td>
<td>$\overline{\mid} 6 - 7 - 3 0$</td>
</tr>
<tr>
<td>$-7x^2 + 18x$</td>
<td>$-7 + 18$</td>
<td>$\overline{\mid} 6 - 7 - 3 0$</td>
<td></td>
</tr>
<tr>
<td>$(-) -7x^2 + 21x$</td>
<td>$(-) -7 + 21$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-3x + 9$</td>
<td>$-3 + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(-) -3x + 9$</td>
<td>$(-) -3 + 9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can use the synthetic division shown in the example above to outline a procedure for synthetic division of any polynomial by a binomial.

Key Concept  Synthetic Division Algorithm

To divide a polynomial by the factor $x - c$, complete each step.

Step 1 Write the coefficients of the dividend in standard form.

Step 2 Multiply the first coefficient by $c$. Write the product under the second coefficient.

Step 3 Add the product and the second coefficient.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example

Divide $6x^3 - 25x^2 + 18x + 9$ by $x - 3$.

\[
\begin{array}{c|c c c c}
3 & 6 & -25 & 18 & 9 \\
\mid & \overline{\mid} & 18 & -21 & -9 \\
6 & -7 & -3 & 0 \\
\overline{\mid} & \downarrow & = \text{Add terms.} & \Rightarrow = \text{Multiply by } c, \text{ and write the product.}
\end{array}
\]
As with division of polynomials by long division, remember to use zeros as placeholders for any missing terms in the dividend. When a polynomial is divided by one of its binomial factors \( x - c \), the quotient is called a **depressed polynomial**.

### Example 4. Synthetic Division

**Divide using synthetic division.**

**a.** \( (2x^4 - 5x^2 + 5x - 2) \div (x + 2) \)

Because \( x + 2 = x - (-2) \), \( c = -2 \). Set up the synthetic division as follows, using zero as a placeholder for the missing \( x^3 \)-term in the dividend. Then follow the synthetic division procedure.

\[
\begin{array}{c|ccccc}
-2 & 2 & 0 & -5 & 5 & -2 \\
\hline & & & & & \\
\end{array}
\]

The quotient has degree one less than that of the dividend, so

\[
\frac{2x^4 - 5x^2 + 5x - 2}{x + 2} = 2x^3 - 4x^2 + 3x - 1.
\]

**b.** \( (10x^3 - 13x^2 + 5x - 14) \div (2x - 3) \)

Rewrite the division expression so that the divisor is of the form \( x - c \).

\[
\frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = (10x^3 - 13x^2 + 5x - 14) \cdot \frac{1}{2x - 3}
\]

So, \( c = \frac{3}{2} \). Perform the synthetic division.

\[
\begin{array}{c|cccc}
\frac{3}{2} & 10 & -13 & 5 & -14 \\
\hline & & & & \\
\end{array}
\]

So, \( \frac{10x^3 - 13x^2 + 5x - 14}{2x - 3} = 5x^2 + x + 4 - \frac{1}{x - \frac{3}{2}} \) or \( 5x^2 + x + 4 - \frac{2}{2x - 3} \).

**Guided Practice**

4A. \( (4x^3 + 3x^2 - x + 8) \div (x - 3) \)

4B. \( (6x^4 + 11x^3 - 15x^2 - 12x + 7) \div (3x + 1) \)

### 2 The Remainder and Factor Theorems

When \( d(x) \) is the divisor \( (x - c) \) with degree 1, the remainder is the real number \( r \). So, the division algorithm simplifies to

\[
f(x) = (x - c) \cdot q(x) + r.
\]

Evaluating \( f(x) \) for \( x = c \), we find that

\[
f(c) = (c - c) \cdot q(c) + r = 0 \cdot q(c) + r \text{ or } r.
\]

So, \( f(c) = r \), which is the remainder. This leads us to the following theorem.

**KeyConcept: Remainder Theorem**

If a polynomial \( f(x) \) is divided by \( x - c \), the remainder is \( r = f(c) \).
The Remainder Theorem indicates that to evaluate a polynomial function \( f(x) \) for \( x = c \), you can divide \( f(x) \) by \( x - c \) using synthetic division. The remainder will be \( f(c) \). Using synthetic division to evaluate a function is called **synthetic substitution**.

### Real-World Example 5 Use the Remainder Theorem

**FOOTBALL** The number of tickets sold during the Northside High School football season can be modeled by \( t(x) = x^3 - 12x^2 + 48x + 74 \), where \( x \) is the number of games played. Use the Remainder Theorem to find the number of tickets sold during the twelfth game of the Northside High School football season.

To find the number of tickets sold during the twelfth game, use synthetic substitution to evaluate \( t(x) \) for \( x = 12 \).

\[
\begin{array}{c|cccc}
  12 & 1 & -12 & 48 & 74 \\
  & 12 & 0 & 576 \\
  \hline
  1 & 0 & 48 & 650 \\
\end{array}
\]

The remainder is 650, so \( t(12) = 650 \). Therefore, 650 tickets were sold during the twelfth game of the season.

**CHECK** You can check your answer using direct substitution.

\[
t(x) = x^3 - 12x^2 + 48x + 74
\]

\[
t(12) = (12)^3 - 12(12)^2 + 48(12) + 74 \text{ or } 650
\]

You can use synthetic division to perform this test.

### Guided Practice

5. **FOOTBALL** Use the Remainder Theorem to determine the number of tickets sold during the thirteenth game of the season.

If you use the Remainder Theorem to evaluate \( f(x) \) at \( x = c \) and the result is \( f(c) = 0 \), then you know that \( c \) is a zero of the function and \( (x - c) \) is a factor. This leads us to another useful theorem that provides a test to determine whether \( (x - c) \) is a factor of \( f(x) \).

### Key Concept Factor Theorem

A polynomial \( f(x) \) has a factor \( (x - c) \) if and only if \( f(c) = 0 \).

You can use synthetic division to perform this test.

### Example 6 Use the Factor Theorem

Use the Factor Theorem to determine if the binomials given are factors of \( f(x) \). Use the binomials that are factors to write a factored form of \( f(x) \).

a. \( f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3; (x - 1), (x + 3) \)

Use synthetic division to test each factor, \( (x - 1) \) and \( (x + 3) \).

\[
\begin{array}{c|cccc}
  1 & 4 & 21 & 25 & -5 & 3 \\
  & 4 & 25 & 50 & 45 \\
  \hline
  1 & 0 & 25 & 50 & 45 \\
\end{array}
\]

Because the remainder when \( f(x) \) is divided by \( (x - 1) \) is 48, \( f(1) = 48 \) and \( (x - 1) \) is not a factor.

\[
\begin{array}{c|cccc}
  -3 & 4 & 21 & 25 & -5 & 3 \\
  & -12 & -27 & 6 & -3 \\
  \hline
  4 & 9 & -2 & 1 & 0 \\
\end{array}
\]

Because the remainder when \( f(x) \) is divided by \( (x + 3) \) is 0, \( f(-3) = 0 \) and \( (x + 3) \) is a factor.

Because \( (x + 3) \) is a factor of \( f(x) \), we can use the quotient of \( f(x) \div (x + 3) \) to write a factored form of \( f(x) \).

\[
f(x) = (x + 3)(4x^3 + 9x^2 - 2x + 1)
\]
Technology Tip

Zeros You can confirm the zeros on the graph of a function by using the zero feature on the CALC menu of a graphing calculator.

CHECK If \((x + 3)\) is a factor of \(f(x) = 4x^4 + 21x^3 + 25x^2 - 5x + 3\), then \(-3\) is a zero of the function and \((-3, 0)\) is an x-intercept of the graph. Graph \(f(x)\) using a graphing calculator and confirm that \((-3, 0)\) is a point on the graph.

b. \(f(x) = 2x^3 - x^2 - 41x - 20; (x + 4), (x - 5)\)

Use synthetic division to test the factor \((x + 4)\).

\[
\begin{array}{c|cccc}
-4 & 2 & -1 & -41 & -20 \\
 & & -8 & 36 & 20 \\
\hline
 & 2 & -9 & -5 & 0 \\
\end{array}
\]

Because the remainder when \(f(x)\) is divided by \((x + 4)\) is 0, \(f(-4) = 0\) and \((x + 4)\) is a factor of \(f(x)\).

Next, test the second factor, \((x - 5)\), with the depressed polynomial \(2x^2 - 9x - 5\).

\[
\begin{array}{c|cc}
5 & 2 & -9 \\
 & 10 & 5 \\
\hline
 & 2 & 1 \\
\end{array}
\]

Because the remainder when the quotient of \(f(x) \div (x + 4)\) is divided by \((x - 5)\) is 0, \(f(5) = 0\) and \((x - 5)\) is a factor of \(f(x)\).

Because \((x + 4)\) and \((x - 5)\) are factors of \(f(x)\), we can use the final quotient to write a factored form of \(f(x)\).

\[f(x) = (x + 4)(x - 5)(2x + 1)\]

CHECK The graph of \(f(x) = 2x^3 - x^2 - 41x - 20\) confirms that \(x = -4, x = 5,\) and \(x = -\frac{1}{2}\) are zeros of the function.

Guided Practice

Use the Factor Theorem to determine if the binomials given are factors of \(f(x)\). Use the binomials that are factors to write a factored form of \(f(x)\).

6A. \(f(x) = 3x^3 - x^2 - 22x + 24; (x - 2), (x + 5)\)

6B. \(f(x) = 4x^3 - 34x^2 + 54x + 36; (x - 6), (x - 3)\)

You can see that synthetic division is a useful tool for factoring and finding the zeros of polynomial functions.

Concept Summary

Synthetic Division and Remainders

If \(r\) is the remainder obtained after a synthetic division of \(f(x)\) by \((x - c)\), then the following statements are true.

- \(r\) is the value of \(f(c)\).
- If \(r = 0\), then \((x - c)\) is a factor of \(f(x)\).
- If \(r = 0\), then \(c\) is an x-intercept of the graph of \(f\).
- If \(r = 0\), then \(x = c\) is a solution of \(f(x) = 0\).
Factor each polynomial completely using the given factor and long division. (Example 1)

1. \(x^3 + 2x^2 - 23x - 60; x + 4\)
2. \(x^3 + 2x^2 - 21x + 18; x - 3\)
3. \(x^3 + 3x^2 - 18x - 40; x - 4\)
4. \(4x^3 + 20x^2 - 8x - 96; x + 3\)
5. \(-3x^3 + 15x^2 + 108x - 540; x - 6\)
6. \(6x^3 - 7x^2 - 29x - 12; 3x + 4\)
7. \(x^4 + 12x^3 + 38x^2 + 12x - 63; x^2 + 6x + 9\)
8. \(x^4 - 3x^3 - 36x^2 + 68x + 240; x^2 - 4x - 12\)

Divide using long division. (Examples 2 and 3)

9. \((5x^4 - 3x^3 + 6x^2 - x + 12) ÷ (x - 4)\)
10. \((x^6 - 2x^3 + x^2 - 3x^2 + 3x - x + 24) ÷ (x + 2)\)
11. \((4x^4 - 8x^3 + 12x^2 - 6x + 12) ÷ (2x + 4)\)
12. \((2x^4 - 7x^3 + 38x^2 + 103x + 60) ÷ (x - 3)\)
13. \((6x^6 - 6x^5 + 6x^4 - 15x^3 + 2x^2 + 10x - 6) ÷ (2x - 1)\)
14. \((108x^5 - 36x^4 + 75x^3 + 36x + 24) ÷ (3x + 2)\)
15. \((x^4 + x^3 + 6x^2 + 18x - 216) ÷ (x^2 - 3x^2 + 18x - 54)\)
16. \((4x^4 + 14x^3 + 110x - 84) ÷ (2x^2 + x - 12)\)
17. \(6x^3 - 12x^2 + 10x - 2x^2 - 8x + 8\)
18. \(3x^3 + 2x + 3\)
19. \(12x^3 + 5x^4 - 15x^3 + 19x^2 - 4x - 28\)
20. \(3x^3 + 2x^2 - x + 6\)

Divide using synthetic division. (Example 4)

21. \((x^4 - x^3 + 3x^2 - 6x - 6) ÷ (x - 2)\)
22. \((2x^4 + 4x^3 - 2x^2 + 8x - 4) ÷ (x + 3)\)
23. \((3x^4 - 9x^3 - 24x - 48) ÷ (x - 4)\)
24. \((x^3 - 3x^3 + 6x^2 + 9x + 6) ÷ (x + 2)\)
25. \((12x^5 + 10x^4 - 18x^3 - 12x^2 - 8) ÷ (2x - 3)\)
26. \((36x^4 - 6x^3 + 12x^2 - 30x - 12) ÷ (3x + 1)\)
27. \((45x^5 + 6x^4 + 3x^3 + 8x + 12) ÷ (3x - 2)\)
28. \((48x^5 + 28x^4 + 68x^3 + 11x + 6) ÷ (4x + 1)\)
29. \((60x^6 + 78x^5 + 9x^4 - 12x^3 - 25x - 20) ÷ (5x + 1)\)
30. \((16x^6 - 56x^5 - 24x^4 + 96x^3 - 42x^2 - 30x + 105) ÷ 2x - 7\)

Educational use The number of U.S. students, in thousands, that graduated with a bachelor’s degree from 1970 to 2006 can be modeled by \(y(x) = 0.0002x^5 - 0.016x^4 + 0.512x^3 - 7.15x^2 + 47.52x + 800.27\), where \(x\) is the number of years since 1970. Use synthetic substitution to find the number of students that graduated in 2005. Round to the nearest thousand. (Example 5)

Find each \(f(x)\) using synthetic substitution. (Example 5)

31. \(f(x) = 4x^5 - 3x^4 + x^3 - 6x^2 + 8x - 15; c = 3\)
32. \(f(x) = 3x^6 - 2x^5 + 4x^4 - 2x^3 + 8x - 3; c = 4\)
33. \(f(x) = 2x^6 + 5x^5 - 3x^4 + 6x^3 - 9x^2 + 3x - 4; c = 5\)
34. \(f(x) = 4x^6 + 8x^5 - 6x^3 - 5x^2 + 6x - 4; c = 6\)
35. \(f(x) = 10x^5 + 6x^4 - 8x^2 - 7x^2 + 3x + 8; c = 6\)
36. \(f(x) = -6x^7 + 4x^5 - 8x^4 + 12x^3 - 15x^2 - 9x + 64; c = 2\)
37. \(f(x) = -2x^8 + 6x^5 - 4x^4 + 12x^3 - 6x + 24; c = 4\)

Use the Factor Theorem to determine if the binomials given are factors of \(f(x)\). Use the binomials that are factors to write a factored form of \(f(x)\). (Example 6)

38. \(f(x) = x^4 - 2x^3 - 9x^2 + x + 6; (x + 2), (x - 1)\)
39. \(f(x) = x^4 + 2x^3 - 5x^2 + 8x + 12; (x - 1), (x + 3)\)
40. \(f(x) = x^4 - 2x^3 + 24x^2 + 18x + 135; (x - 5), (x + 5)\)
41. \(f(x) = 3x^4 - 22x^3 + 13x^2 + 11x - 40; (3x - 1), (x - 5)\)
42. \(f(x) = 4x^4 - x^3 - 36x^2 - 11x + 30; (4x - 1), (x - 6)\)
43. \(f(x) = 3x^4 - 35x^3 + 38x^2 + 56x + 64; (3x - 2), (x + 2)\)
44. \(f(x) = 5x^5 + 38x^4 - 68x^2 + 59x + 30; (5x - 2), (x + 8)\)
45. \(f(x) = 4x^5 - 9x^4 + 39x^3 + 24x^2 + 75x + 63; (4x + 3), (x - 1)\)

46. TREES The height of a tree in feet at various ages in years is given in the table.

<table>
<thead>
<tr>
<th>Age</th>
<th>Height</th>
<th>Age</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.3</td>
<td>24</td>
<td>73.8</td>
</tr>
<tr>
<td>6</td>
<td>13.8</td>
<td>26</td>
<td>82.0</td>
</tr>
<tr>
<td>10</td>
<td>23.0</td>
<td>28</td>
<td>91.9</td>
</tr>
<tr>
<td>14</td>
<td>42.7</td>
<td>30</td>
<td>101.7</td>
</tr>
<tr>
<td>20</td>
<td>60.7</td>
<td>36</td>
<td>111.5</td>
</tr>
</tbody>
</table>

a. Use a graphing calculator to write a quadratic equation to model the growth of the tree.
b. Use synthetic division to evaluate the height of the tree at 15 years.

47. BICYCLING Patrick is cycling at an initial speed \(v_0\) of 4 meters per second. When he rides downhill, the bike accelerates at a rate \(a\) of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use \(d(t) = v_0t + \frac{1}{2}at^2\) to find how long it will take Patrick to ride down the hill, where \(d(t)\) is distance traveled and \(t\) is given in seconds.
Factor each polynomial using the given factor and long division. Assume \( n > 0 \).

48. \( x^{3n} + x^{2n} - 14x^n - 24; x^n + 2 \)
49. \( x^{3n} + x^{2n} - 12x^n + 10; x^n - 1 \)
50. \( 4x^{3n} + 2x^{2n} - 10x^n + 4; 2x^n + 4 \)
51. \( 9x^{3n} + 24x^{2n} - 171x^n + 54; 3x^n - 1 \)

52. MANUFACTURING An 18-inch by 20-inch sheet of cardboard is cut and folded into a bakery box.

53. \( x^3 - kx^2 + 2x - 4 \)
54. \( x^3 + 18x^2 + kx + 4 \)
55. \( x^3 + 4x^2 - kx + 1 \)
56. \( 2x^3 - x^2 + x + k \)

57. SCULPTING Esteban will use a block of clay that is 3 feet by 4 feet by 5 feet to make a sculpture. He wants to reduce the volume of the clay by removing the same amount from the length, the width, and the height.

a. Write a polynomial function to model the situation.
b. Graph the function.
c. He wants to reduce the volume of the clay to \( \frac{3}{5} \) of the original volume. Write an equation to model the situation.
d. How much should he take from each dimension?

Use the graphs and synthetic division to completely factor each polynomial.

58. \( f(x) = 8x^4 + 26x^3 - 103x^2 - 156x + 45 \) (Figure 2.3.1)
59. \( f(x) = 6x^5 + 13x^4 - 153x^3 + 54x^2 + 724x - 840 \) (Figure 2.3.2)

60. MULTIPLE REPRESENTATIONS In this problem, you will explore the upper and lower bounds of a function.

a. GRAPHICAL Graph each related polynomial function, and determine the greatest and least zeros. Then copy and complete the table.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Greatest Zero</th>
<th>Least Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 - 2x^2 - 11x + 12 )</td>
<td>( x^2 + 6x^3 + 3x^2 - 10x )</td>
<td>( x^2 - x^3 - 2x^2 )</td>
</tr>
</tbody>
</table>

b. NUMERICAL Use synthetic division to evaluate each function in part a for three integer values greater than the greatest zero.

c. VERBAL Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for an integer greater than its greatest zero.

d. NUMERICAL Use synthetic division to evaluate each function in part a for three integer values less than the least zero.

e. VERBAL Make a conjecture about the characteristics of the last row when synthetic division is used to evaluate a function for a number less than its least zero.

61. CHALLENGE Is \( (x - 1) \) a factor of \( 18x^{165} - 15x^{135} + 8x^{105} - 15x^{35} + 47 \)? Explain your reasoning.

62. WRITING IN MATH Explain how you can use a graphing calculator, synthetic division, and factoring to completely factor a fifth-degree polynomial with rational coefficients, three integral zeros, and two non-integral, rational zeros.

63. REASONING Determine whether the statement below is true or false. Explain.

If \( h(y) = (y + 2)(3y^2 + 11y - 4) - 1 \), then the remainder of \( h(y) \) divided by \( y + 2 \) is \(-1\).

64. \( \frac{x^3 + kx^2 - 34x + 56}{x + 7} \)

65. \( \frac{x^6 + kx^4 - 8x^3 + 173x^2 - 16x - 120}{x - 1} \)

66. \( \frac{kx^3 + 2x^2 - 22x - 4}{x - 2} \)

67. CHALLENGE If \( 2x^2 - dx + (31 - d^2)x + 5 \) has a factor \( x - d \), what is the value of \( d \) if \( d \) is an integer?

68. WRITING IN MATH Compare and contrast polynomial division using long division and using synthetic division.
Determine whether the degree $n$ of the polynomial for each graph is even or odd and whether its leading coefficient $a_n$ is positive or negative. (Lesson 2–2)

72. SKYDIVING The approximate time $t$ in seconds that it takes an object to fall a distance of $d$ feet is given by $t = \sqrt{\frac{d}{16}}$. Suppose a skydiver falls 11 seconds before the parachute opens. How far does the skydiver fall during this time period? (Lesson 2–1)

73. FIRE FIGHTING The velocity $v$ and maximum height $h$ of water being pumped into the air are related by $v = \sqrt{2gh}$, where $g$ is the acceleration due to gravity (32 feet/second$^2$). (Lesson 1–7)
   a. Determine an equation that will give the maximum height of the water as a function of its velocity.
   b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department's needs? Explain.

Solve each system of equations algebraically. (Lesson 0–5)

74. $5x - y = 16$
   $2x + 3y = 3$

75. $3x - 5y = -8$
   $x + 2y = 1$

76. $y = 6 - x$
   $x = 4.5 + y$

77. $2x + 5y = 4$
   $3x + 6y = 5$

78. $7x + 12y = 16$
   $5y - 4x = -21$

79. $4x + 5y = -8$
   $3x - 7y = 10$

82. REVIEW The first term in a sequence is $x$. Each subsequent term is three less than twice the preceding term. What is the 5th term in the sequence?
   A $8x - 21$
   B $8x - 15$
   C $16x - 39$
   D $16x - 45$
   E $32x - 43$
   F $8x - 15$
   G $16x - 45$
   H $32x - 43$
   I $8x - 21$
   J $8x - 15$
   K $16x - 39$

83. Use the graph of the polynomial function. Which is not a factor of $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?
   A $(x - 2)$
   B $(x + 2)$
   C $(x - 1)$
   D $(x + 1)$
   E $(x - 2)$
   F $(x + 2)$
   G $(x - 1)$
   H $(x + 1)$
   I $(x - 2)$
   J $(x + 2)$
   K $(x - 1)$
   L $(x + 1)$

f(x) = x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4